STAT240 Problem Set 4

Due April 6th in class

Regular problems:

1. Define $W_{c,k}(p,q) = \inf_{\pi \in \Pi(p,q)} \mathbb{E}_{(x,y)\sim\pi}[c(x,y)^k]^{1/k}$. For $k \geq 1$, show that $W_{c,k}$ is a metric if c is a metric.

2. Define the set of mth moment resilient distributions as

$$\mathcal{G}^{W_1}_{\mathsf{mth}} = \{p \mid |\mathbb{E}_r[\langle x, v\rangle^m] - \mathbb{E}_p[\langle x, v\rangle^m]| \leq \rho \text{ whenever } r \text{ is } \epsilon\text{-friendly under } \langle x, v\rangle^m \text{ and } \|v\|_2 = 1\}. \quad (1)$$

   For even $m \geq 2$, show that $\mathcal{G}^{W_1}_{\mathsf{mth}}$ has small modulus of continuity under the loss

$$L(p, T) = \sup_{\|v\|_2 \leq 1} |\mathbb{E}_{x\sim p}[\langle x, v\rangle^m] - \langle T, v^{\otimes m}\rangle|. \quad (2)$$

3. Suppose that p has kth moments bounded by σ. For even $m \geq 2$ and $k \geq m$, show that p is mth moment resilient with

$$\rho = \mathcal{O}(1)^m \cdot \mathcal{O}(\epsilon^m + \sigma^{\frac{k(m-1)}{k-1}} \epsilon^{\frac{k-m}{k-1}}). \quad (3)$$

4. Suppose we define an alternative second moment resilient set as

$$\mathcal{G}^{W_1}_{\mathsf{sec}'} = \{p \mid |\mathbb{E}_r[\langle x, v\rangle^2] - \mathbb{E}_p[\langle x, v\rangle^2]| \leq \rho \text{ whenever } r \text{ is } \epsilon\text{-friendly under } |\langle x, v\rangle| \text{ with } \|v\|_2 = 1\}. \quad (4)$$

   Here the difference is we consider friendliness under $|\langle x, v\rangle|$ instead of $\langle x, v\rangle^2$. Show that $\mathcal{G}^{W_1}_{\mathsf{sec}'}$ still has small modulus of continuity.

5. Suppose that $L(p, \theta)$ can be represented as $L(p, \theta) = \sup_{f \in \mathcal{F}_\theta} \mathbb{E}_{x\sim p}[f(x)] - L^*(f, \theta)$. Define the two conditions ($\downarrow$) and ($\uparrow$) as

$$\mathbb{E}_r[f(x)] - L^*(f, \theta^*(p)) \leq \rho_1 \text{ for all } f \in \mathcal{F}_{\theta^*(p)} \text{ and } r \text{ that are } \epsilon\text{-friendly under } f, \quad (\downarrow)$$

   and

$$L(p, \theta) \leq \rho_2 \text{ if } \forall f \in \mathcal{F}_\theta \text{ there is an } r \text{ that is } \epsilon\text{-friendly under } f \text{ with } \mathbb{E}_r[f(x)] - L^*(f, \theta) \leq \rho_1. \quad (\uparrow)$$

   Let $\mathcal{G}^{W_1}_L(\rho_1, \rho_2, \epsilon)$ be the family of distributions satisfying ($\downarrow$) and ($\uparrow$). Show that $\mathcal{G}^{W_1}_L(\rho_1, \rho_2, \epsilon)$ has small modulus of continuity.

Challenge problems (turn in as a separate document typset in LaTeX):

6. Consider linear regression with $L(p, \theta) = \mathbb{E}_{(X,Y)\sim p}[(Y - \langle \theta, X\rangle)^2]$ (note this is now the *risk* rather than the *excess risk* that we had before). Let $X' = [X, Y]$ be the d-dimensional vector that concatenates X and Y, and define $Z = Y - \langle \theta^*(p), X\rangle$. Suppose the following two conditions hold:

$$\mathbb{E}_p[|\langle v, X'\rangle|^3] \leq \kappa^3 \mathbb{E}[\langle v, X'\rangle^2]^2 \text{ for all } \|v\|_2 = 1, \quad (5)$$

$$\mathbb{E}_p[Z^2] \leq \sigma^2. \quad (6)$$

   Assuming that $\kappa\epsilon\psi^{-1}(4\kappa/\epsilon) \leq \frac{1}{8}$, show that this family of distributions has modulus of continuity that is $\mathcal{O}(\sigma^2 + \epsilon^2)$.

   [You may want to do this by first showing that the distribution is resilient in the sense of problem 5.*]*

7. Let p be the uniform distribution on $[-1, 1]^d$. If p_n is the empirical distribution over n samples, show that $\mathbb{E}[W_1(p, p_n)] \geq \Omega(n^{-1/d})$.