Verifying Stochastic Systems

Jacob Steinhardt and Russ Tedrake

Massachusetts Institute of Technology

Motivation

- Robots are often subject to large uncertainties
  - dynamical: wind gusts
  - perceptual: stereo vision
- To maximize performance, want to plan against typical case (99%) rather than worst case (100%)

Lyapunov equations

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- Sufficient condition for stability: non-negative function V such that

$$\dot{V}(x) \leq 0$$

Martingale condition

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$$\mathbb{E}[\dot{V}(x)] \leq 0$$

Martingale condition

$$\underbrace{\mathbb{E}[\dot{V}(x)]}_{\displaystyle \overbrace{\lim_{\Delta t \to 0^+} \frac{\mathbb{E}[V(x(t+\Delta t)) \mid x(t)] - V(x(t))}{\Delta t}}} \leq 0$$

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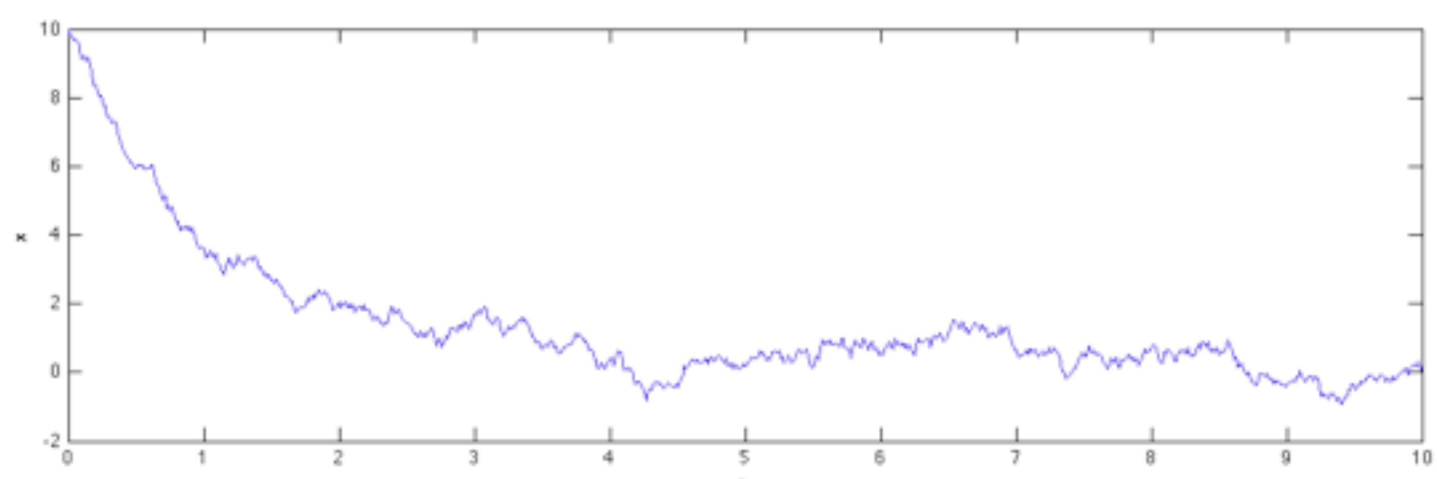
$$dx(t) = -xdt + dw(t)$$

- What is $\mathbb{E}[\dot{V}(0)]$?



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- todo: choose and optimize over a family of functions V

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- $E[dV/dt] = p(x)\exp(x^T J x)$

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\begin{aligned}
& p(x)e^{x^T Jx} \leq c \\
\iff \quad & p(x) \leq ce^{-x^T Jx} \\
\Longleftarrow \quad & p(x) \leq c(1 - x^T Jx)
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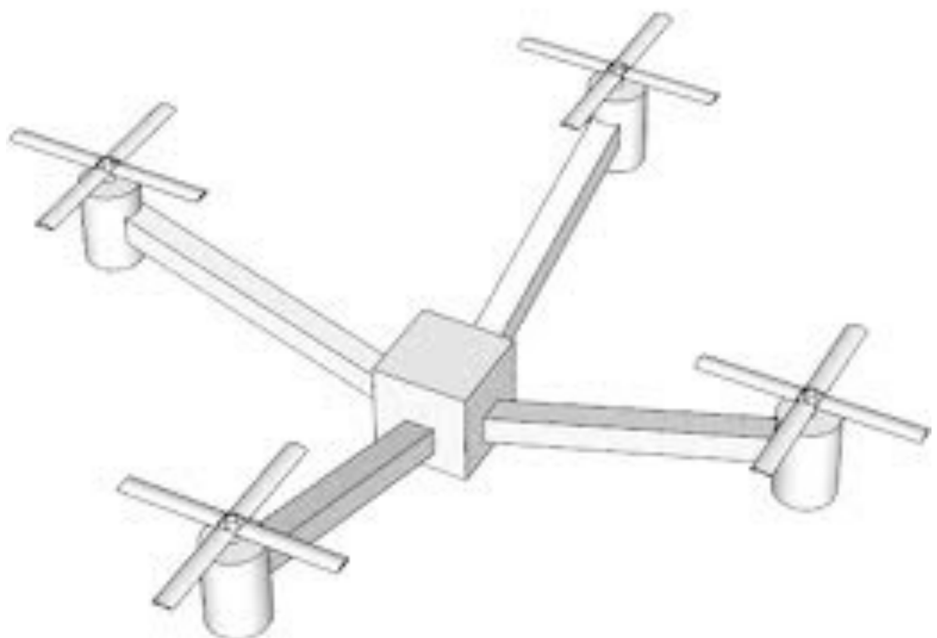
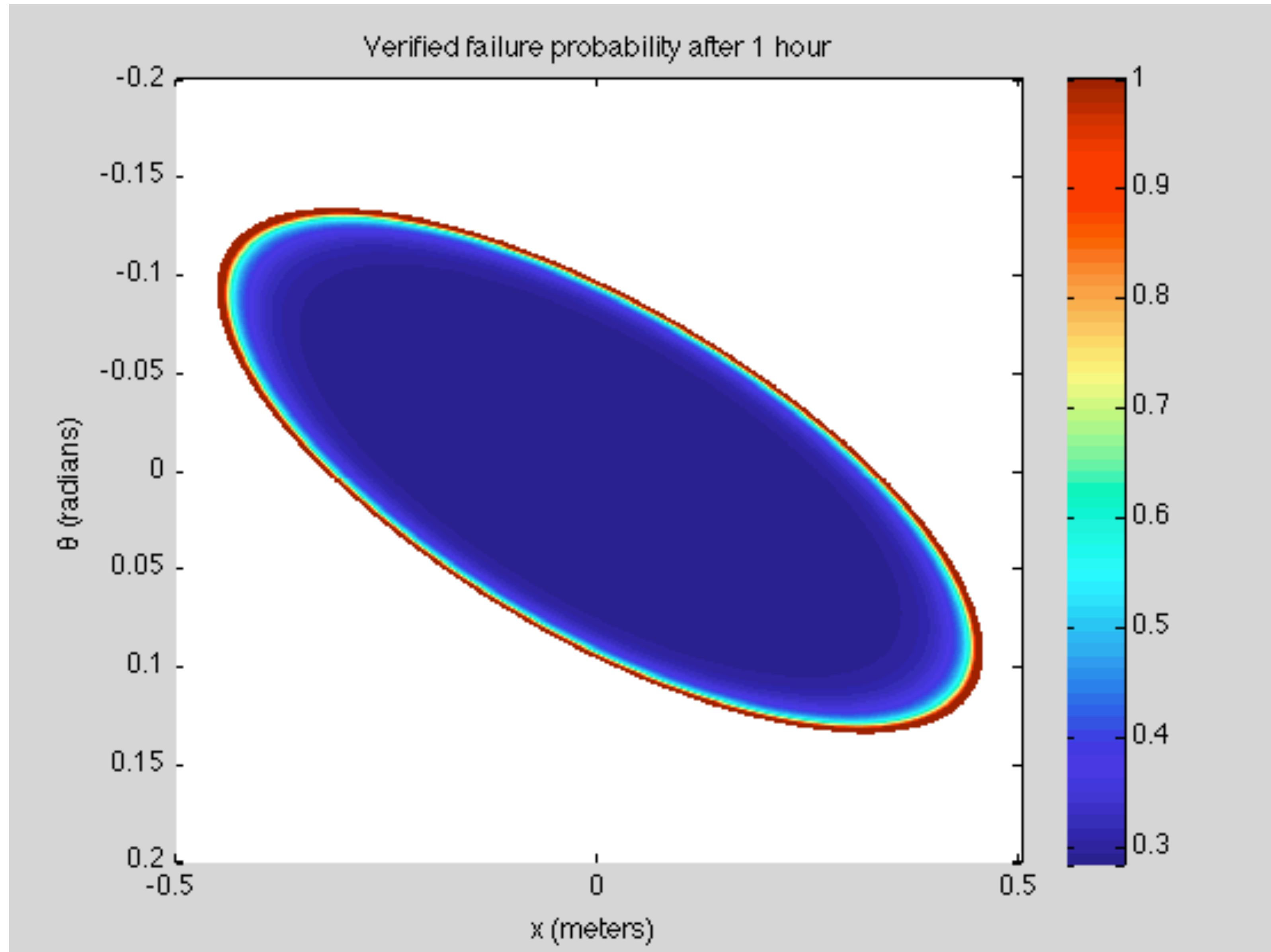
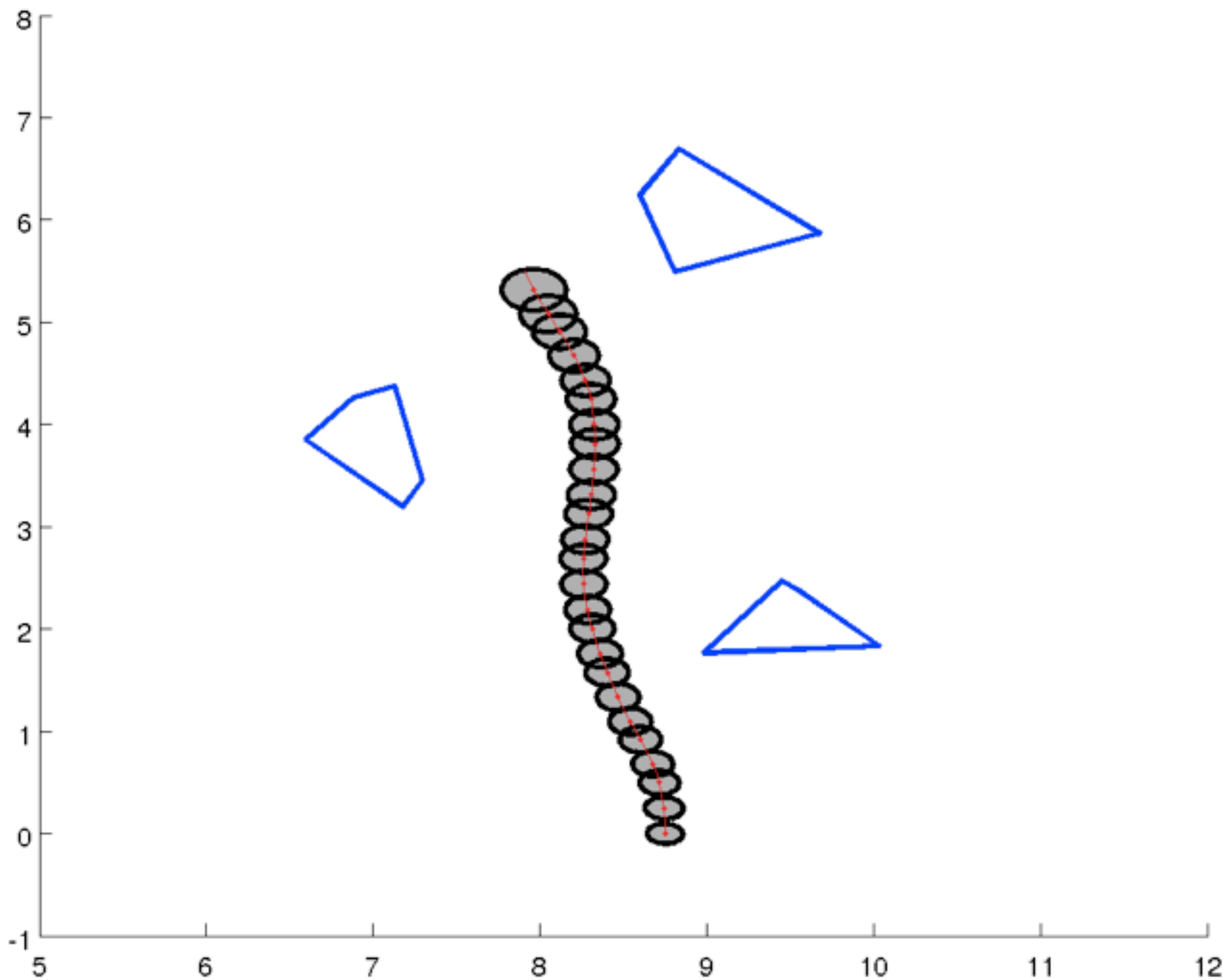
- polynomial condition: optimize with SOS programming

Results: Quadrotor





Results: UAV



Further reading

- http://groups.csail.mit.edu/locomotion/
- LQR-trees tutorial on Friday in the "Integrated Planning and Control" workshop