# Permutations with Ascending and Descending Blocks

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## Definition

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## Question

How many permutations in $S_n$ have a given descent set $S$ and lie in a given conjugacy class $\mathcal{C}$?

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### Example

The 6 $(2, 2)$-ascending permutations are

$$\begin{array}{cc} 12|34 & 13|24 \\ 14|23 & 23|14 \\ 24|13 & 34|12 \end{array}$$

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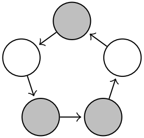
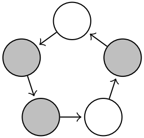
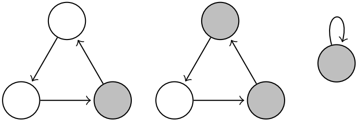
## Example

(a) and (b) are the same $(2, 3)$-compatible ornament. (c) is $(3, 4)$-compatible.

(a)  (b)  (c) 

### Theorem (Gessel and Reutenauer, 1993)

*The* $(a_1, \ldots, a_k)$*-ascending permutations are in bijection with* $(a_1, \ldots, a_k)$*-compatible ornaments where every cycle is aperiodic. This bijection preserves cycle structure.*

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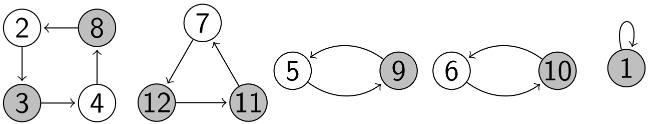
## Example

$(a_1, a_2) = (7, 5)$, $\pi = 1\ 3\ 4\ 8\ 9\ 10\ 12\ |2\ 5\ 6\ 7\ 11$



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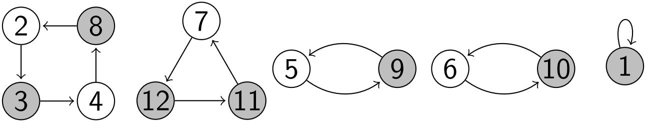


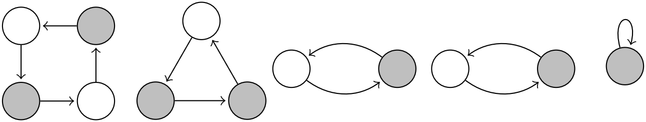

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$(a_1, a_2) = (8, 10)$, $S = \{1\}$
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### Question

Can we generalize the Gessel-Reutenauer bijection to $(a_1, \ldots, a_k, S)$-permutations?

### Theorem

*There is an injection from the* $(a_1, \ldots, a_k, S)$*-permutations to the* $(a_1, \ldots, a_k)$*-compatible ornaments. This injection preserves cycle structure.*

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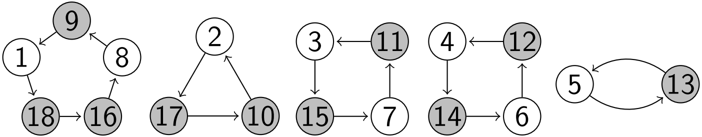

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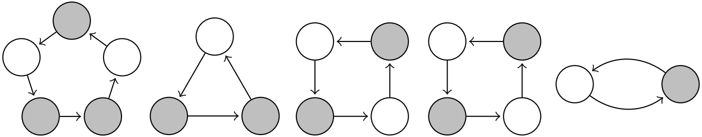

**Corollary (Conjectured in (Eriksen, Freij, Wästlund, 2007))**

*For any permutation $\sigma$ of $\{1, \ldots, k\}$ and conjugacy class $\mathcal{C}$ of $S_n$, the $(a_1, \ldots, a_k, S)$-permutations in $\mathcal{C}$ are in bijection with the $(a_{\sigma(1)}, \ldots, a_{\sigma(k)}, \sigma(S))$-permutations in $\mathcal{C}$.*

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### Corollary (Theorem 7.1 in (Gessel and Reutenauer, 1993))

*If $\mathcal{C}$ satisfies certain mild properties, then the number of elements of $\mathcal{C}$ with descent set $D$ equals the number of elements of $\mathcal{C}$ with descent set $\{1, \ldots, n-1\} \setminus D$.*

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### Corollary (Theorem 7.2 in (Gessel and Reutenauer, 1993))

*The number of involutions with descent set $D$ equals the number of involutions with descent set $\{1, \ldots, n-1\} \setminus D$.*

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### Question

The Gessel-Reutenauer bijection implies that there are $\binom{a_1+\ldots+a_k}{a_1,\ldots,a_k}$ $A$-compatible ornaments such that every cycle is aperiodic. Is there a simpler proof of this fact?

Thank you.


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